Large deviations of additive observables in simple interacting particle systems: equilibrium & non-equilibrium

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Classical and quantum dynamics

What one gains from forgetting probabilities and turning to the quantum world
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- Correspondence
  - generator of **stochastic** classical system
  - Hamiltonian of **quantum** XXZ chain

(Well known at least in the stat. mech. community.)
Classical and quantum dynamics

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- Use: dictionary between
  - regimes of large deviations of dynamical (i.e. additive) observables
  - phases across a Quantum Phase Transition
Classical and quantum dynamics

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  - generator of *stochastic* classical system
  - Hamiltonian of *quantum* XXZ chain

(Well known at least in the stat. mech. community.)

- **Use:** dictionary between
  - regimes of *large deviations* of *dynamical* (*i.e.* additive) observables
  - phases across a Quantum Phase Transition

- **Perspectives opened; questions raised**
  - finite-size effects
  - large-/small-scale spectrum

[I will ask questions to *you.*]
Exclusion Processes – generic settings

- Configurations: occupation numbers \( \{n_i\} \)
- Exclusion rule: \( 0 \leq n_i \leq N \)
- Markov evolution for the probability \( P(\{n_i\}, t) \)
  \[
  \partial_t P(\{n_i\}, t) = \sum_{n_i'} \left[ W(n_i' \to n_i) P(\{n_i'\}, t) - W(n_i \to n_i') P(\{n_i\}, t) \right]
  \]
- Large deviation function of “additive” observables \( A \)
  \[
  \langle e^{-sA} \rangle \sim e^{t \psi(s)}
  \]

\( A = \) total current \( Q \) on time window \([0, t]\)
\( A = \) total activity \( K \) on time window \([0, t]\)
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  \langle e^{-sA} \rangle \sim e^{t \psi(s)}
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  \( (\Leftrightarrow \text{determining } P(A, t)) \)
  \[
  A = \text{total current } Q \text{ on time window } [0, t] = \# \text{jumps} - \text{jumps}
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  \[
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\( A = \) total current \( Q \) on time window \([0, t]\)

\( A = \) total activity \( K \) on time window \([0, t]\)

\( A = \# \text{jumps} - \text{jumps} \)

\( A = \# \text{jumps} + \text{jumps} \)
Operator representation

\[\text{Evolution of probability vector } P: \]
\[
\partial_t P = \mathbb{W} P
\]
\[
\mathbb{W} = \sum_{1 \leq k \leq L-1} \left[ S^+_k S^-_{k+1} + S^-_k S^+_ {k+1} - \hat{n}_k \hat{n}_{k+1} - \hat{n}_{k+1} \hat{n}_k \right]
\]
\[
\quad + \alpha \left[ S^+_1 - \hat{n}_1 \right] + \gamma \left[ S^-_1 - \hat{n}_1 \right]
\]
\[
\quad + \delta \left[ S^+_L - \hat{n}_L \right] + \beta \left[ S^-_L - \hat{n}_L \right]
\]
\[
[\hat{n} = N - \hat{n}]
\]
\[S^\pm = S^x \pm iS^y \text{ and } S^z = \hat{n} - \frac{N}{2} \text{ are spin operators (of “spin” } j = \frac{N}{2}\text{)}
\]
Operator representation

\[ \rho_0 \quad \rho_1 \]

Evolution of probability vector \( P \):

\[ \partial_t P = \mathbb{W} P \]

\[ \mathbb{W} = \sum_{1 \leq k \leq L-1} \left[ S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k \hat{n}_{k+1} - \hat{n}_{k+1} \hat{n}_k \right] + \alpha \left[ S_1^+ - \hat{n}_1 \right] + \gamma \left[ S_1^- - \hat{n}_1 \right] + \delta \left[ S_L^+ - \hat{n}_L \right] + \beta \left[ S_L^- - \hat{n}_L \right] \]

\[ S^\pm = S^x \pm i S^y \quad \text{and} \quad S^z = \hat{n} - \frac{N}{2} \quad \text{are spin operators (of “spin” } j = \frac{N}{2} \text{)} \]

densities \( \rho_0 = \frac{\alpha}{\alpha + \gamma} \); \( \rho_1 = \frac{\delta}{\delta + \beta} \); contact rates \( a_0 = \frac{\alpha}{\gamma} \); \( a_1 = \frac{\delta}{\beta} \)
Evolution of probability vector $P$:

$$\partial_t P = \mathbb{W} P$$

$$\mathbb{W} = \sum_{1 \leq k \leq L-1} \left[ S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k \hat{n}_{k+1} - \hat{n}_{k+1} \hat{n}_k \right]$$

$$+ \alpha \left[ S_1^+ - \hat{n}_1 \right] + \gamma \left[ S_1^- - \hat{n}_1 \right]$$

$$+ \delta \left[ S_L^+ - \hat{n}_L \right] + \beta \left[ S_L^- - \hat{n}_L \right]$$

$$[\hat{n} = N - \hat{n}]$$

$S^\pm = S^x \pm iS^y$ and $S^z = \hat{n} - \frac{N}{2}$ are spin operators (of “spin” $j = \frac{N}{2}$)

**XXX spin chain Hamiltonian** (up to boundary terms and constants).
Operator representation for **large deviations**

\[
\langle e^{-sK} \rangle \sim e^{t\psi(s)} \quad \text{with} \quad \psi(s) = \max \text{Sp} \ W_s
\]

\[
W_s = \sum_{1 \leq k \leq L-1} \left[ e^{-s} S_k^+ S_{k+1}^- + e^{-s} S_k^- S_{k+1}^+ - \hat{n}_k \hat{n}_{k+1} - \hat{n}_{k+1} \hat{n}_k \right] \\
+ \alpha \left[ e^{-s} S_1^+ - \hat{n}_1 \right] + \gamma \left[ e^{-s} S_1^- - \hat{n}_1 \right] \\
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\]

for the **activity** \( K: \) **XXZ spin chain Hamiltonian**
Operator representation for large deviations

\[ \langle e^{-sQ} \rangle \sim e^{t\psi(s)} \quad \text{with} \quad \psi(s) = \max S \mathbb{W}_s \]

\[ \mathbb{W}_s = \sum_{1 \leq k \leq L-1} \left[ e^{sS_k^+} S_{k+1}^- + e^{-sS_k^-} S_{k+1}^+ - \hat{n}_k \hat{n}_{k+1} - \hat{n}_{k+1} \hat{n}_k \right] \]

\[ + \alpha \left[ e^{-sS_1^+} - \hat{n}_1 \right] + \gamma \left[ e^{sS_1^-} - \hat{n}_1 \right] \]

\[ + \delta \left[ e^{sS_L^+} - \hat{n}_L \right] + \beta \left[ e^{-sS_L^-} - \hat{n}_L \right] \]

for the current \( Q \): “asymmetric” XXZ spin chain Hamiltonian
Example 1: use of rotational symmetry
map non-equilibrium current fluctuations
to equilibrium current fluctuations
Mapping non-eq to eq

[Imparato, VL, van Wijland, PTPS 184 276]

Large deviations of the current

\[ \psi(s) = \max \text{Sp} \ W(s) \]

\[
W(s) = \sum_{1 \leq k \leq L-1} \langle \hat{S}_k \cdot \hat{S}_{k+1} \rangle + \text{constant} \\
+ \alpha [S_1^+ - \hat{n}_1] + \gamma [S_1^- - \hat{n}_1] \\
+ \delta [S_L^+ e^s - \hat{n}_L] + \beta [S_L^- e^{-s} - \hat{n}_L]
\]
Mapping non-eq to eq

[Imparato, VL, van Wijland, PTPS 184 276]

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\[ + \delta [S_L^+ e^s - \hat{n}_L] + \beta [S_L^- e^{-s} - \hat{n}_L] \]

Local transformation

\[ Q^{-1} W(s) Q = \sum_{1 \leq k \leq L-1} \vec{S}_k \cdot \vec{S}_{k+1} \]

\[ + \alpha' [S_1^+ - \hat{n}_1] + \gamma' [S_1^- - \hat{n}_1] \]

\[ + \delta' [S_L^+ e^{s'} - \hat{n}_L] + \beta' [S_L^- e^{-s'} - \hat{n}_L] \]

describes contact with reservoirs of same densities
SO(3) symmetry [Imparato, VL, van Wijland, PTPS 184 276]

Detailed transformation: (on one site)

\[ Q = 1 + xS^x - iyS^y + zS^z \] (invertible)

performs a rotation of the vector \( S = (S^x, S^y, S^z) \) of spin operators

\[ Q^{-1}S^xQ = (RS)_1 \quad Q^{-1}S^yQ = (RS)_2 \quad Q^{-1}S^zQ = (RS)_3 \]

for some SO(3) rotation matrix \( R \).
SO(3) symmetry  

[Imparato, VL, van Wijland, PTPS 184 276]

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for some SO(3) rotation matrix \( R \).

Form of the matrix: (Cayley form)

\[ R = (I + A)(I - A)^{-1} \]

\[ A = \begin{pmatrix} 0 & -iz & y \\ iz & 0 & -ix \\ -y & ix & 0 \end{pmatrix} \]
Large deviations

[Imparato, VL, van Wijland, PTPS 184 276]

Result: (transforming all sites)

\[ Q^{-1} \mathbb{W}_{\text{res}}(s; \rho_0, \rho_1; a_0, a_1) Q = \mathbb{W}_{\text{res}}(s'; \rho_0', \rho_1'; a_0, a_1) \]

with “primed” variables

\[
\rho_0' = \frac{(1 + x) \rho_0 - x - z}{1 - x} \\
\rho_1' = (x + e^{-s} - z(1 - e^{-s})) \frac{[x + e^s + z(1 - e^s)] \rho_1 - x - z}{1 - x^2} \\
e^{-s'} = \frac{x + e^{-s} + z(e^{-s} - 1)}{1 + xe^{-s} + z(e^{-s} - 1)}
\]
Symmetric exclusion process

System in equilibrium

\[ \rho_0 \]

\[ \rho' \]

Local transformation

\[ \text{Prob}_{\text{non-eq}}^{\text{stationn.}} \text{ (current)} \]

\[ \text{Prob}_{\text{eq}}^{\text{stationn.}} \text{ (current')} \]
**Probabilistic interpretation**

Measure $\hat{P}(n, s, t)$ biased by $e^{-sQ}$

**Mapping:**

$$\hat{P}(n, s, t; \rho_0, \rho_1; a_0, a_1) = \langle n | e^{tW(s; \rho_0, \rho_1; a_0, a_1)} | P_{\text{init}} \rangle$$

$$= \langle n | Q e^{tW(s'; \rho_0', \rho_1'; a_0, a_1)} Q^{-1} | P_{\text{init}} \rangle$$

- new projection state
- new initial condition
Probabilistic interpretation

Measure $\hat{P}(n, s, t)$ biased by $e^{-sQ}$

Mapping: $\hat{P}(n,s, t; \rho_0, \rho_1; a_0, a_1) = \langle n | e^{tW(s; \rho_0, \rho_1; a_0, a_1)} | P_{\text{init}} \rangle = \langle n | Q e^{tW(s'; \rho_0', \rho_1'; a_0, a_1)} Q^{-1} | P_{\text{init}} \rangle$

new projection state new initial condition

Question: What is the mathematical embedding (in terms of process & prob.)? (Duality, Radon-Nykodym? caveat: prob. not preserved)

Generalization:

★ higher dimensions
★ generic network and current
★ more than two reservoirs
★ see also: Derrida & Gerschenfeld (ω variable)

Akkermans, Bodineau, Derrida & Shpielberg (1d LDF for $d > 1$)
Example 2: exclusion process on a ring
Focus on a simple situation

**Simple** exclusion process (SSEP): max. occupation $N = 1$; spins $S \rightarrow \sigma$

Periodic boundary conditions
Focus on a simple situation

**Simple** exclusion process (SSEP): max. occupation $N = 1$; spins $S \leftrightarrow \sigma$

Periodic boundary conditions
Fixed total particle number $N_0$

Density: $\rho_0 = N_0 / L$

Ring geometry
Focus on a simple situation \( s \leftrightarrow \text{activity } K \)

**Simple** exclusion process (SSEP): max. occupation \( N = 1 \); spins \( S \leftrightarrow \sigma \)

Periodic boundary conditions

Fixed total particle number \( N_0 \)

Density: \( \rho_0 = N_0/L \)

\[
W_s = \sum_{k=1}^{L-1} \left[ e^{-s} \left( \sigma_k^+ \sigma_{k+1}^- + \sigma_k^+ \sigma_{k+1}^- \right) - \hat{n}_k(1 - \hat{n}_{k+1}) - (1 - \hat{n}_k)\hat{n}_{k+1} \right] \\
= \frac{L - 1}{2} - \frac{e^{-s}}{2} \mathcal{H} \Delta \\
\mathcal{H} \Delta = -\sum_{k=1}^{L-1} \left[ \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \right] \quad \text{with} \quad \Delta = e^s
## Classical/Quantum dictionary

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<th>Quantum Spin Chain</th>
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<td><strong>counting factor</strong> $\Delta = e^s$ of the activity $K$</td>
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<td>[ E_L(s) = \min \text{Sp} , \mathbb{H}_\Delta ]</td>
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Bethe Ansatz

[Appert, Derrida, VL, van Wijland, PRE 78 021122]
Bethe Ansatz

Coordinate Bethe Ansatz: Integrability known from long ; difficulty: $L \to \infty$

- eigenvector of components

$$\sum_{\mathcal{P}} A(\mathcal{P}) \prod_{i=1}^{N_0} [\zeta_{\mathcal{P}(i)}]^{x_i}$$

- eigenvalue

$$\psi(s) = -2N_0 + e^{-s}[\zeta_1 + \ldots + \zeta_{N_0}] - e^{-s} \left[ \frac{1}{\zeta_1} + \ldots + \frac{1}{\zeta_{N_0}} \right]$$

- Bethe equations

$$\zeta_i^L = \prod_{j=1}^{N_0} \left[ -\frac{1 - 2e^s \zeta_i + \zeta_i \zeta_j}{1 - 2e^s \zeta_j + \zeta_i \zeta_j} \right]$$
Bethe Ansatz

[Appert, Derrida, VL, van Wijland, PRE 78 021122]

Repartition of Bethe roots in the complex plane

- ●: finite-size solution
- ○: infinite-size limit
Finite-size effects

- large deviation function

\[ \psi(s) = -2L\rho_0(1 - \rho_0)s + L^{-2}\mathcal{F}(u) + \ldots \quad \text{with} \quad u = L^2\rho_0(1 - \rho_0)s \]

- universal function (singular in \( u = \frac{\pi^2}{2} \))

\[ \mathcal{F}(u) = \sum_{k \geq 2} \frac{(-2u)^kB_{2k-2}}{\Gamma(k)\Gamma(k+1)} \]
Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE 78 021122]

- large deviation function

\[ \psi(s) = -2L\rho_0(1 - \rho_0)s + L^{-2}F(u) + \ldots \quad \text{with} \quad u = L^2\rho_0(1 - \rho_0)s \]

- universal function (singular in \( u = \frac{\pi^2}{2} \))
Finite-size effects

- Large deviation function
  \[ \psi(s) = -2L\rho_0(1 - \rho_0)s + \frac{L^{-2}F(u)}{\rho_0} + \ldots \quad \text{with} \quad u = L^2\rho_0(1 - \rho_0)s \]

- Universal function (singular in \( u = \frac{\pi^2}{2} \))

![Non-analyticity](image)

Non-analyticity \( F(u) \) dynamical phase transition at \( s_c = \frac{\pi^2}{2L^2\rho_0(1 - \rho_0)}u \)
Macroscopic limit

A reminder: propagator in quantum mechanics

\[ \langle \text{final} \mid e^{itH} \mid \text{initial} \rangle \]
Macroscopic limit

A reminder: propagator in quantum mechanics

\[
\langle \text{final} \mid e^{it\mathcal{H}} \mid \text{initial} \rangle = \int dz_1 \ldots dz_n \langle \text{final} \mid e^{i\Delta t\mathcal{H}} \mid z_n \rangle \langle z_{n-1} \mid e^{i\Delta t\mathcal{H}} \mid z_{n-2} \rangle \ldots \\
\ldots \langle z_1 \mid e^{i\Delta t\mathcal{H}} \mid \text{initial} \rangle \\
= \int \mathcal{D}p\mathcal{D}q \exp\{i\frac{1}{\hbar}S[p, q]\}
\]

where \( p = p(x, t) \) and \( q = q(x, t) \) are generically space- & time-dependent fields.

“semi-classical limit” recovered in the large \( \frac{1}{\hbar} \) limit [saddle-point]
Macroscopic limit

For exclusion processes using \(SU(2)\) coherent states:

\[
\langle \rho_f | e^{tW} | \rho_i \rangle = \int_{\rho(0)=\rho_i} \mathcal{D}\rho \mathcal{D}\hat{\rho} \exp\{L \mathcal{S}[\hat{\rho}, \rho]\}
\]
Macroscopic limit

For exclusion processes

Using \( SU(2) \) coherent states:

\[
\langle \rho_f | e^{t\mathcal{W}} | \rho_i \rangle = \int_{\rho(0)=\rho_i}^{\rho(t)=\rho_f} \mathcal{D}\rho\mathcal{D}\hat{\rho} \ exp\{L S[\hat{\rho}, \rho]\} \\
\langle e^{-sK} \rangle \sim \langle \rho_f | e^{t\mathcal{W}_s} | \rho_i \rangle = \int_{\rho(0)=\rho_i}^{\rho(t)=\rho_f} \mathcal{D}\rho\mathcal{D}\hat{\rho} \ exp\{L S_s[\hat{\rho}, \rho]\}
\]

Again: use saddle-point to handle the large \( L \) limit.
Macroscopic limit

For exclusion processes

Same $S_s[\hat{\rho}, \rho]$ as the MSR action of the Langevin evolution:

$$\partial_t \rho(x, t) = -\partial_x \left[ -\partial_x \rho(x, t) + \xi(x, t) \right]$$

$$\langle \xi(x, t) \xi(x', t') \rangle = \frac{1}{L} \rho(x, t) (1 - \rho(x, t)) \delta(x' - x) \delta(t' - t)$$

One recovers the action of fluctuating hydrodynamics $[L \to \infty]$  

[Spohn; Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim]
Macroscopic approach

Hydrodynamic limit

Macroscopic limit

[Tailleur, Kurchan, VL, JPA 41 505001]

For exclusion processes

Same $S_s[\hat{\rho}, \rho]$ as the MSR action of the Langevin evolution:

$$\partial_t \rho(x, t) = -\partial_x \left[ -\partial_x \rho(x, t) + \xi(x, t) \right]$$

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One recovers the action of fluctuating hydrodynamics \[ L \rightarrow \infty \]

[Spohn; Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim]

And obtains non-trivial finite-size corrections \[ \text{lattice contribs.} \]

(those affecting the saddle, not the fluctuations around it)
$\psi(s)$: again

[Appert, Derrida, VL, van Wijland, PRE 78 021122]

Periodic boundary conditions
More general fluctuating hydrodynamics

\[
\frac{1}{L_t} \langle Q \rangle \propto D(\rho) \quad \text{(Fourier's law)}
\]

\[
\frac{1}{L_t} \langle Q^2 \rangle_c = \sigma(\rho) \quad \text{(For the SSEP, } \sigma(\rho) = \rho(1 - \rho))
\]
$\psi(s)$: again 

[Appert, Derrida, VL, van Wijland, PRE 78 021122]

Periodic boundary conditions
More general fluctuating hydrodynamics

$$\frac{1}{L_t} \langle Q \rangle \propto D(\rho)$$

(Fourier’s law)

$$\frac{1}{L_t} \langle Q^2 \rangle_c = \sigma(\rho)$$

(For the SSEP, $\sigma(\rho) = \rho(1 - \rho)$)

Saddle point evaluation

$$\langle e^{-sK} \rangle \sim \int \mathcal{D} \rho \mathcal{D} \hat{\rho} \exp\{L S_s[\hat{\rho}, \rho]\}$$
\( \psi(s) \): again

[Appert, Derrida, VL, van Wijland, PRE 78 021122]

Periodic boundary conditions
More general fluctuating hydrodynamics

\[
\frac{1}{L_t} \langle Q \rangle \propto D(\rho) \quad \text{(Fourier’s law)}
\]

\[
\frac{1}{L_t} \langle Q^2 \rangle_c = \sigma(\rho) \quad \text{(For the SSEP, } \sigma(\rho) = \rho(1 - \rho)\text{)}
\]

Saddle point evaluation

\[
\langle e^{-sK} \rangle \sim \int \mathcal{D}\rho \mathcal{D}\hat{\rho} \, \exp\{L S_s[\hat{\rho}, \rho]\}
\]

Large deviation function

[assuming uniform profile \( \rho(x) = \rho \)]

\[
\psi(s) = -s \frac{\langle K \rangle_c}{t} \quad \text{at saddle-point}
+ L^{-2} D \mathcal{F}(u) \quad \text{with} \quad u = L^2 s \frac{\sigma(\rho_0) \sigma''(\rho_0)}{8D^2}
\]

\int \text{of quadratic fluctuations}
Correspondence between the (Gaussian) integration of small fluctuations AND discreteness of Bethe root repartition.

More general?
Correspondence between the (Gaussian) integration of small fluctuations AND discreteness of Bethe root repartition.

More general?

Repartition of Bethe roots for $s > s_c$?
Correspondence between
the (Gaussian) integration of small fluctuations
AND
discreteness of Bethe root repartition.

More general?

Repartition of Bethe roots for $s > s_c$?

Fluctuating hydrodynamics for quantum chains?
Dynamical phase transition \([VL, Garrahan, van Wijland, JPA 45 175001]\)

Rescaling of the large deviation function \([\text{singularity at } \lambda_c > 0 \text{ as } L \rightarrow \infty]\)

\[\varphi(\lambda) = \lim_{L \rightarrow \infty} L \psi(\lambda/L^2)\]

Using the correct non-uniform saddle-point profile for \(\lambda > \lambda_c\)

\[\lambda_c = \frac{\pi^2}{\sigma(\rho_0)}\]
(can be large!)

\(\varphi(\lambda)\)

\(\lambda_c\)

non-uniform profile

uniform profile
Dynamical phase transition [VL, Garrahan, van Wijland, JPA 45 175001]

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see also: for LDF of $Q$
[Bodineau, Derrida, PRE 78 021122]
phase transition in WASEP for large dev. (non-stationary profile)
[Jona-Lasinio et al.]
generic criterion for instability
Dynamical phase transition \cite{VL, Garrahan, van Wijland, JPA 45 175001}

Optimal saddle-point profile $\rho(x)$

Increasing $\lambda$ ($\lambda > \lambda_c$)
<table>
<thead>
<tr>
<th><strong>SSEP</strong></th>
<th><strong>Quantum Spin Chain</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>local occupation number $n_k \ (1 \leq k \leq L)$</td>
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<td>XY degenerate groundstate ($\Delta = 0$)</td>
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Sketch of derivation [VL, Garrahan, van Wijland, JPA 45 175001]

Saddle-point equations for the profile $\rho(x)$ take the form

$$(\partial_x \rho(x))^2 + E_P(\rho(x)) = 0$$
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“Oscillations” depict the non-uniform profile $\rho(x)$
Excitations

What about solutions with *more than one* kink + anti-kink?

\[ \varphi(\lambda) \]

- \[ \lambda_c \]
- \[ 4\lambda_c \]

Corresponding profiles \( \rho(x) \)
Small sizes: the ground state

Aim: experimental realizations with cold atoms
→ non-periodic (but isolated, 1D) system
→ smaller sizes & finite-temperature & excited state

\[ \varphi(\lambda) \]
Small sizes: the full spectrum

$L = 9$ sites
$N_0 = 3$ particles
Small sizes: the full spectrum

$L = 9$ sites
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LDF symmetries and quantum mechanics
Bonn – Dec 10th 2015
Small sizes: the full spectrum

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$N_0 = 3$ particles

infinite-size ground state
infinite-size excited states
Small sizes: the full spectrum

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gathering(?) of microscopic eigenvalues $\rightarrow$ macroscopic ($L = \infty$) states
Summary

Microscopic approach:
- operator formalism
- XXZ spin chain
- Bethe Ansatz

Macroscopic approach:
- MFT, saddle-point method, dynamical phase transition

Questions:
- Finite-size crossover around a quantum phase transition? Between:
  - Luttinger Liquid ($s \neq 1$)
  - Phase-separated ferromagnet ($s \neq 1 + 1$)
- Across the transition: continuum spectrum! gaped spectrum?
- XXZ transition not at $\Delta = 1$ but at $\Delta = 1 + O(L^2)$
- Are scaling exponents/functions known? Are they interesting?
- Hydrodynamics approaches for quantum questions?
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- Non-Hermitian operators $\leftrightarrow$ dissipation in Lindblad?
Thank you for your attention!

References:

★ Marc Cheneau, Vivien Lecomte et al. 
  work in progress (2014–)

★ Vivien Lecomte, Juan P. Garrahan, Frédéric van Wijland 

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  PTPS 184 276 (2010)

★ Cécile Appert-Rolland, Bernard Derrida, Vivien Lecomte, 
  Frédéric van Wijland 