Thermally Induced Magnetic Switching in Thin Ferromagnetic Annuli

Kirsten Martens\textsuperscript{a}, D.L. Stein\textsuperscript{b}, and A.D. Kent\textsuperscript{c}

\textsuperscript{a} Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, 69120
Heidelberg, Germany
\textsuperscript{b} Department of Physics, University of Arizona, 1118 E. 4th St., Tucson, AZ 85721, USA
\textsuperscript{c} Department of Physics, New York University, 4 Washington Place, NY 10003, USA

ABSTRACT

We consider magnetization reversal due to thermal fluctuations in thin, submicron-scale rings. These mesoscopic ferromagnetic particles are of particular interest as potential information storage components in magnetoelectronic devices, because their lack of sharp ends result in a magnetization density that is significantly more stable against reversal than in thin needles and other geometries. Their two-dimensional nature and rotational symmetry allow us to incorporate long-range magnetostatic forces in a fully analytic treatment, which is not possible in most geometries. We uncover a type of ‘phase transition’ between different activation regimes as magnetic field is varied at fixed ring size. Previous studies of such transitions in classical activation behavior have found that they occur as length is varied, which cannot be realized easily or continuously for most systems. However, the different activation regimes in a single mesoscopic ferromagnet should be experimentally observable by changing the externally applied magnetic field, and by tuning this field the transition region itself can be studied in detail.

Keywords: Magnetization reversal, nanomagnets, magnetic rings, micromagnetics, Néel-Brown theory, Kramers theory, thermal fluctuations, magnetic switching, stochastic escape

1. INTRODUCTION

Understanding the complex dynamical behavior of mesoscopic-scale magnets is important from the point of view of both basic physics and technological applications: magnetoelectronic devices relying on such magnets are used extensively for information storage and other recording purposes.\cite{1, 2} For the stochastic systems community, the problem of thermally induced switching in single-domain systems provides a rich and challenging array of fundamental questions, opportunities for application, and tests of ideas and methods.

Modern investigations into stochastic magnetization reversal go back more than half a century. The classical theory was worked out by Néel\textsuperscript{3} and Brown,\textsuperscript{4} who modelled thermally induced reversal by assuming a spatially uniform magnetization and uniaxial anisotropy. Such an approach is expected to apply to single-domain systems with simple geometries, such as spheres. Relatively recent experimental confirmation of the Néel-Brown theory has been provided\textsuperscript{5} for some systems (15-30 nm Ni, Co, and Dy nanoparticles).

The assumptions behind the Néel-Brown theory presumably break down, however, in more complicated geometries, such as elongated particles or thin films. In such situations one expects spatial variation of the magnetization density,\textsuperscript{6, 9} requiring a new approach. Indeed, it has been observed\textsuperscript{10} that magnetic systems in such geometries exhibit far lower coercivities than predicted by Néel-Brown. An initial step in understanding this effect was provided by Braun,\textsuperscript{6} who studied magnetic reversal in infinitely long cylindrical magnets.\textsuperscript{11} By proposing a new and interesting transition state whose magnetization density varied along the cylindrical axis, he was able to account qualitatively for the greatly reduced coercivity.
However, there are several difficulties with this approach. In general, a serious impediment to analytic (and to a lesser extent, numerical) treatment of stochastic reversal in micro- and nanomagnets arises from long-range forces originating from classical interactions among the magnetic dipoles distributed throughout the system. The energy corresponding to these forces is typically called the magnetostatic self-energy,\textsuperscript{12} and usually leads to nonlocal terms in the effective Hamiltonian of the system. Because Braun’s result was based on approximations neglecting this energy (or more accurately, incorporating it into local anisotropy terms), it was criticized by Aharoni,\textsuperscript{13} who pointed out that the neglected nonlocal magnetostatic energy contributions were significantly larger than some of the included terms, invalidating the result.

A second issue is that for submicron-scale magnets with large aspect ratio, finite system effects are likely to play an important role. In the case of the cylinder, simulations\textsuperscript{7,14} indicate that magnetization reversal in cylindrical-shaped particles proceeds via propagation and coalescence of magnetic ‘end caps’, nucleated at the cylinder ends. (See Braun\textsuperscript{15} for a response to both of these objections.)

In this paper we consider a different geometry that avoids these difficulties: a mesoscopic annulus with sufficiently large aspect ratio that it is effectively two-dimensional. Magnetic systems of this type have recently been fabricated and their properties are receiving increasing attention.\textsuperscript{16,17} Their typical radii are between \((10^2 - 10^3)\) nanometers and their thicknesses can be smaller than 10 nm. Most are made of soft magnetic materials (quality factor \(Q \sim O(10^{-2})\)), such as Fe, fcc Co, or permalloy.

Mesoscopic rings are potentially important for both technological and physical reasons. For the former, they offer the possibility of information storage devices that are unusually stable against information degradation from thermally induced magnetic reversal, even at moderately high temperatures. Unlike the cylinder, the micromagnetic ring has no ends where nucleation is easily initiated, making its magnetic state more stable against thermally induced reversal.

For the latter, these systems possess several features that make them extremely interesting from the point of view of stochastic escape theory. We will see that their geometry is such as to allow us, first, to separate out local and nonlocal contributions to the magnetostatic energy; and second (and even more importantly), to show that the nonlocal contributions are an order of magnitude (or more) smaller than the local terms. This clears the way towards a fully analytic theory for thermally induced magnetic reversal. It should be kept in mind, though, that nonlocal terms can still contribute up to 10% of the total magnetic energy, and because this energy governs the exponential dependence of the reversal rate, the quantitative power of such a theory is still limited. However, because an exact analytic solution can be found to the nonlinear equations governing the reversal process, we can understand in depth its qualitative features. Of particular interest is that our solution indicates the occurrence of a crossover from one type of activation regime (which exhibits Arrhenius behavior) to another regime (which is non-Arrhenius), and more importantly, we find that such a crossover can be realized experimentally for the first time.

This crossover shares a number of features with a type of second-order phase transition in the thermal activation rate. This effect was first noted within the purely classical activation context by Maier and Stein (MS),\textsuperscript{18} as system size was varied in a symmetric Ginzburg-Landau double-well \(\phi^4\) potential, and was later shown to apply more generally to asymmetric systems as well. It turns out that this kind of transition (as an external parameter such as system length or external field is varied) in classical noise-induced activation behavior maps to the transition from classical activation to quantum tunneling as temperature is varied.\textsuperscript{21–32}

In models studied up until now the predicted transition depended on system size, which is not readily varied, and so we are not aware of any instances in which this transition has been observed in any physical system. In the present case, however, the transition depends on two parameters: the system size and the strength of the applied magnetic field. Although the former cannot be continuously varied, the latter can, facilitating experimental tests of the predicted transition. A transition in activation behavior as magnetic field is varied for fixed system size is a central prediction of this paper.

A shorter paper\textsuperscript{33} containing the main features of the analysis presented here can be found at cond-mat/0410561.
2. THE MODEL

The use of a mesoscopic ferromagnetic ring as a recording particle can be envisioned in a simple scenario by considering an electric current running along the $\hat{z}$-direction through the center, with $\hat{z}$ the direction normal to the annulus plane. The current generates a magnetic field in the circumferential direction $\hat{\theta}$. Because the magnetic bending length $\lambda$ (typically of order 10 nm) is much smaller than the typical ring circumference, there are two oppositely polarized stable states, each with magnetization vector pointing everywhere along the circumferential direction. They are degenerate in the absence of an external magnetic field, but of course the applied field breaks the degeneracy. By switching the direction of the current, the relative stability of the two states is also switched. The metastable state can be relatively long-lived if the opposing field is not too great, but will eventually reverse due to thermal fluctuations.

To model this situation, we consider an annulus of thickness $t$, inner radius $R_1$ and outer radius $R_2 = R_1 + \Delta R$, with $t \ll \Delta R \ll R_1$. As described above, consider a circumferential magnetic field $H_\theta = H(r)\hat{\theta}$ produced by a constant current passing through the center of the annulus in the $\hat{z}$-direction. Because $H(r) \sim 1/r$, the variation of field strength with radius over the annulus is $O(\Delta R/R_1) \ll 1$. This allows us to treat the system in a quasi-1D fashion, where the main qualitative features of the problem can already be elucidated. In subsequent work we will drop the requirement that $\Delta R \ll R_1$, in which case a fuller 2D analysis is required.

However, the primary difficulty confronting a fully analytic treatment arises from terms introduced by long-range magnetostatic forces (see, for example, Aharoni\cite{Aharoni96}), as discussed in the preceding section. Nonlocal terms arising from these forces are often of the same (or even larger) order as local terms; in such cases a numerical treatment of thermal activation and reversal is required.\cite{Aharoni96} We will see in the next section, however, that in thin ring geometries of the type under consideration here, nonlocal terms can be an order of magnitude or more smaller than local terms, allowing for a fully analytic treatment.

As will be discussed below, the main effect of magnetostatic forces in these geometries is to produce strong anisotropies, forcing the magnetization vector to lie in the plane and with a preferred boundary orientation parallel to the inner and outer circumferences. The problem can therefore be treated in a quasi-1D fashion in which the magnetization configuration varies only along the $\hat{\theta}$-direction. The system geometry and coordinates are shown in Fig. 1.

Suppose now that the system is initially in its metastable state; i.e., with magnetization vector $\mathbf{M} = -M_0\hat{\theta}$. We wish to determine the rate for thermal fluctuations to reverse the magnetization to its stable direction. We
where magnetization dynamics are governed by the Landau-Lifshitz-Gilbert equation
\[ \partial_t \mathbf{M} = -\gamma [\mathbf{M} \times \mathbf{H}_{\text{eff}}] + (\alpha/M_0) [\mathbf{M} \times \partial_t \mathbf{M}], \tag{1} \]
where \(M_0\) is the (fixed) magnitude of \(\mathbf{M}\), \(\alpha\) the damping constant, and \(\gamma > 0\) the gyromagnetic ratio. The effective field \(\mathbf{H}_{\text{eff}} = -\delta E/\delta \mathbf{M}\) is the variational derivative of the total energy \(E\), which (with free space permeability \(\mu_0 = 1\)) is given by: \[ E[\mathbf{M}(\mathbf{x})] = \frac{\lambda}{2} \int_{\Omega} \left| \nabla \mathbf{M} \right|^2 + \frac{1}{2} \int_{\mathbb{R}^3} d^3x |\nabla U|^2 \]
where \(\Omega\) is the region occupied by the ferromagnet, \(\lambda\) is the exchange length, and \(U\) (defined over all space) satisfies \(\nabla^2 U = \nabla \cdot \mathbf{M}\) along with appropriate boundary conditions. The nonlocal bulk term in Eq. (2), if of the same order as the other (local) terms, would prevent a fully analytical treatment of thermally induced reversal. However, the quasi-2D nature of the problem allows a significant simplification, as shown by Kohn and Slaskikov (KS). Their asymptotic scaling analysis applies to the thin annulus geometry if the aspect ratio \(k = t/R\) and the normalized exchange length \(l = \lambda/R\) are sufficiently small, and \(l^2 \sim k|\log k|\). These constraints restrict the range of ring geometries to which our analysis applies.

We begin by recasting the energy in dimensionless form. Let \(R = (R_1 + R_2)/2\), and define the dimensionless length \(X = x/R\) (and similarly for all other lengths). The thinness of the ring compared to the exchange length allows us to regard magnetization variations as constant along the \(\hat{z}\) direction. Then, integrating along this direction, we have for the bending plus Zeeman energy contribution
\[ \frac{E_b + E_z}{M_0^2 R^3} = kl^2 \int_0^1 \omega d^2 \left[ \left( \nabla \times \mathbf{m} \right)^2 - \frac{H_e \cdot \mathbf{m}}{M_0 l^2} \right], \tag{3} \]
where \(\mathbf{m} = \mathbf{M}/M_0\) is the normalized magnetization and \(\omega\) represents the 2D surface with boundary \(\partial \omega\).

We will find it convenient to use a locally varying coordinate system in which the angle \(\phi' = R\theta\) (cf. Fig. (1)) measures the deviation of the local magnetization vector from the local applied field direction; i.e., \(\phi = 0\) indicates that the local magnetization is parallel to the local field, \(\phi = \pi\) indicates that it is antiparallel, and so on. The parameter \(s' = R\theta\) is the arc length along the circumference. After integrating out the radial coordinate the bending plus Zeeman energy becomes
\[ \frac{E_b + E_z}{M_0^2 R^3} = kl^2 \left[ \left( \log R_2 \right)/R_1 \int_0^{2\pi} d\theta \left\{ 1 + \left( \frac{\partial m_r}{\partial \theta} \right)^2 + \left( \frac{\partial m_\theta}{\partial \theta} \right)^2 \right\} \right. \]
\[ + \left. 2 \left( m_r \frac{\partial m_\theta}{\partial \theta} - m_\theta \frac{\partial m_r}{\partial \theta} \right) \right\} - \frac{\Delta R}{R} \int_0^{2\pi} d\theta 2h' \cdot \mathbf{m} \right] \tag{4} \]
where \(h' = H_R/(2M_0 l^2)\), and \(H_R\) is the external magnetic field at \(R\). The term \(m_r \frac{\partial m_\theta}{\partial \theta} - m_\theta \frac{\partial m_r}{\partial \theta}\) is a ‘winding number’ of \(\phi\) with respect to the local direction. For fixed \(M_0\), it does not contribute to the magnetization equations of motion given by Eq. (1).
We are now ready to turn to the magnetostatic energy term. The analysis of KS\textsuperscript{36} can be applied to the present thin-film geometry to show that this contribution asymptotically separates into bulk and surface terms:

\[
\frac{E_{\text{mag}}}{M_{\text{eff}}R^3} = \frac{1}{2}k \int_{\omega} d^2X m_z^2 + (1/4\pi k^2) \log k \int_{\partial\omega} ds (\mathbf{m} \cdot \hat{r})^2 \\
+ \frac{1}{2}k^2 \int_{\omega} d^2X |\nabla \cdot \mathbf{m}|^2 R^{-1/2},
\]

(5)

where the final integral is the squared $H^{-1/2}$ Sobolev norm of $\nabla \cdot \mathbf{m}$. The orders of magnitude $k \sim 10^{-2}$ and $l^2 \sim k \log k \sim 10^{-2} - 10^{-1}$ are presently attainable. Then the first term of Eq. (5) is larger than the others by roughly two orders of magnitude, and so we can safely ignore fluctuations of $\mathbf{m}$ out of the plane. The magnetization vector can therefore be written, in cylindrical coordinates, as $\mathbf{m} = (m_r, m_\theta, m_z) = (\sin \phi, \cos \phi, 0)$.

The second term, like the first, is a magnetostatic surface (or shape anisotropy) term, a local term which will play an important role in the analysis. The third term represents a nonlocal magnetostatic bulk energy. We will find (cf. Sec. 5) that this bulk term (when nonzero) will be an order of magnitude or more smaller than the others, and so to a first approximation\textsuperscript{38} we can ignore it also. Nevertheless, doing so could still result, for some values of ring size and external field, in an error of up to 10\textsuperscript{36} in the computation of the action; as a result we can hope for at best reasonably good quantitative predictions of the logarithm of the escape rate. The important qualitative features uncovered by our analysis should remain unaffected, however.

Finally, subtracting out both the constant terms and the first derivative term (which gives zero contribution), noting that the boundary integral occurs over both inner and outer radii, and rescaling lengths once more gives

\[
\mathcal{E} = \int_0^{\ell/2} ds \left[ \left( \frac{\partial \phi}{\partial s} \right)^2 + \sin^2 \phi - 2h \cos \phi \right],
\]

(6)

In Eq. (6), $\mathcal{E} = E/E_0 = E/[2M_{\text{eff}}^2R^2\Delta R\sqrt{\ell d^2}]$ where $c = (1/2\pi)(k \log k)/l^2)(R/\Delta R)$, $s = \theta \sqrt{c}$, $\ell = 2\sqrt{c}$, and $h = H_R/(2M_0^2c)$. In deriving (6) we used the fact that $\log(R_2/R_1) = \Delta R/R + O[(\Delta R/R)^2]$. The error is negligible for the geometries considered here; for example, with the ring parameters used in Fig. 4, $\Delta R/R = 1/5$ and $\log(R_2/R_1) = .20067$.

The parameter $c$ ($0 < c < \infty$) depends on the ring size and material properties; it represents the ratio of the anisotropy energy scale to the bending energy scale, and determines the width of a Bloch wall.

4. THE TRANSITION IN ACTIVATION BEHAVIOR

Because we are working in a temperature regime where thermal activation over a barrier is the dominant switching mechanism, the reversal rate $\Gamma$ is given by the usual Kramers formula $\Gamma \sim \Gamma_0 \exp(-\Delta W/k_BT)$. The dynamics occurs in an effective potential governed by the Zeeman and shape anisotropy energies, so the activation barrier $\Delta W \gg k_BT$ is just the energy difference between the metastable and ‘saddle’ states. The latter is the configuration of highest energy along the system’s optimal escape path.\textsuperscript{39} More precisely, it is the state of lowest energy with a single negative eigenvalue (and corresponding unstable direction) of the linearized zero-noise dynamics.\textsuperscript{18,19,40} The rate prefactor $\Gamma_0$ is determined by fluctuations about this optimal path.

The magnetization configurations that play a role in determining the switching rate — i.e., the stable, unstable, and saddle states — are all time-independent solutions of the LLG equations. For fixed $M_0$, Eq. (1) and the variational equation $\mathbf{H}_{\text{eff}} = -\delta E/\delta \mathbf{M}$ yield a nonlinear Euler-Lagrange equation that must be satisfied by any such time-independent solution:

\[
d^2 \phi/ds^2 = \sin \phi \cos \phi + h \sin \phi.
\]

(7)

When $0 \leq h < 1$ there are three solutions of Eq. (7) in which $\phi$ is independent of $\theta$; we will call these ‘constant’ configurations. However, it should be noted that they are nonuniform, because in all three $\mathbf{m}$ varies with position. These solutions are the stable state $\phi = 0$ ($\mathbf{m} = \hat{\theta}$); the metastable state $\phi = \pi$ ($\mathbf{m} = -\hat{\theta}$); and a pair
of degenerate unstable states $\phi = \cos^{-1}(-h)$, which constitute the saddle for a range of $(\ell, h)$. The $\phi = 0, \pi$ solutions are degenerate when $h = 0$, and the $\phi = \pi$ solution becomes unstable at $h = 1$.

We have also found a nonconstant, or ‘instanton’, solution of Eq. (7), which is the saddle for the remaining range of $(\ell, h)$. It is

$$\phi(s, m) = 2 \cot^{-1} \left[ \text{dn} \left( \frac{s - s_0}{\delta} \middle| m \right) \frac{\text{sn}(\mathcal{R}|m)}{\text{cn}(\mathcal{R}|m)} \right], \tag{8}$$

where $\text{dn}(\cdot|m)$, $\text{sn}(\cdot|m)$, and $\text{cn}(\cdot|m)$ are the Jacobi elliptic functions\footnote{Accordingly, imposition of the periodic boundary condition yields a relation between $\ell$ and $m$:}

$$\ell = 2K(m)\delta. \tag{11}$$

The limit $m \to 1$ corresponds to $\ell \to \infty$ at fixed $h$. In this limit, Eqs. (8)-(11) reduce to Braun’s solution,\footnote{It was derived for an infinite cylinder. (As noted earlier, for the cylindrical geometry nonlocal magnetostatic forces are large,\footnote{so it is not clear whether this solution provides a good approximation to a transition state.) The opposite limit, $m \to 0$, corresponds to a ‘collapse’ of the instanton solution to the constant state $\phi = \cos^{-1}(-h)$. (This can be seen by noting that $\text{dn}(x|0) = 1$.) The $m \to 0$ limit represents the transition between the two types of activation regime, and the ‘phase boundary’ of this transition is found by relating the length and field at $m = 0$. The equation of the phase boundary is then}

$$\ell_c = \pi \delta_c = \frac{2\pi}{\sqrt{1 - h_c^2}}, \tag{12}$$

and is depicted in Fig. 2.

How do we physically account for the crossover between the two activation regimes? Fig. 2 depicts the following scenario: at fixed $h$, the constant configuration is the saddle for $\ell < \ell_c$ and the instanton is the saddle.
for \(\ell > \ell_c\). This can be understood as follows: at fixed field, as \(\ell\) is lowered the bending energy becomes larger so that at sufficiently small \(\ell\) the constant transition state becomes energetically preferred. (There is a second transition at even smaller \(\ell\), where the bending energy becomes so large that the magnetization lies along a single Euclidean direction everywhere; we do not consider such small length scales here.) Conversely, at fixed \(\ell > 2\pi\) the constant configuration is the saddle for \(h > h_c\), and the nonconstant for \(h < h_c\). At these lengthscales, when \(h > h_c\) the Zeeman term dominates the anisotropy term, favoring the constant configuration (which clearly has lower Zeeman energy than the instanton). It is clear from Fig. 2 that when \(\ell \leq 2\pi\), the bending energy has grown sufficiently large so that the constant configuration is the saddle for all \(h\).

We now examine how this transition affects the reversal rate.

### 5. COMPUTATION OF THE SWITCHING RATE

We noted in Sec. 4 that the exponential dependence of the reversal rate on temperature is given by \(\Delta W\), the energy difference between the metastable and saddle states. In the \((\ell, h)\) region below the phase boundary in Fig. 2, where the transition state is constant, the action difference grows linearly with \(\ell\) (at fixed \(h\)). On crossing over to the other side of the phase boundary, \(\Delta W\) bends over quickly and becomes almost flat (cf. Fig 4 of Stein\(^{19}\)) as \(\ell\) increases at fixed \(h\), because the domain wall width remains essentially constant. In Fig. 3(a) we show the activation energy dependence on \(h\) at fixed \(\ell\) (which is more relevant to experiment than varying \(\ell\) at fixed \(h\)).

We noted in Secs. 2 and 3 that the bulk magnetostatic energy is smaller than the other terms in the magnetization configurations considered here, and that it is this property that enables the above analysis to proceed. We can now do a self-consistency check to determine whether this is indeed the case. For the instanton solution (8) at \(\ell = 7\), an upper bound for the ratio of the bulk magnetostatic to the bending energy varies roughly from 0.05 to 0.1 as \(h\) varies; similar numbers are found for other lengths. So to a reasonable approximation this term can be neglected.

We turn now to determination of the prefactor. This is a more involved calculation,\(^{19}\) and the results are simply summarized here. The prefactor is given by

\[
\Gamma_0 = \frac{1}{2\pi} \sqrt{\frac{\det \Lambda_s}{\det \Lambda_u}} |\lambda_{u,0}|,
\]  

(13)
where $\mathbf{A}_t$ ($\mathbf{A}_u$) is the linearized zero-noise dynamical operator determining time evolution of fluctuations about the stable (saddle) state, and $\mathbf{A}_{0\,0}$ is $\mathbf{A}_u$’s single negative eigenvalue, corresponding to the direction along which the optimal escape trajectory approaches the saddle.

The prefactor calculation is relatively straightforward when the saddle is the pair of constant configurations, because in that case it is possible to compute directly the eigenvalues of the stable and unstable states:

$$\Gamma_0^\dagger = \tau_0^{-1} \frac{(1 - h^2)}{\pi} \frac{\sinh(\sqrt{1 - h\ell/2})}{\sin(\sqrt{1 - h\ell/2})},$$

(14)

where $\tau_0^{-1} = \alpha \gamma E_0/M_0 V(1 + \alpha^2)$, with $V$ the ring volume. $\Gamma_0^\dagger$ diverges at $\ell_c(h)$, or conversely $h_c(\ell)$, as expected; in this limit, $\Gamma_0^\dagger \sim \text{const} \times (\ell - \ell_c)^{-1}$ as $\ell \to \ell_c$ at fixed $h$, or as $(h - h_c)^{-1}$ as $h \to h_c^+$ at fixed $\ell$. The prefactor in this region for fixed $\ell$ as $h$ varies is plotted in Fig. 3(b). The divergence arises from the vanishing of the eigenvalue of a pair of degenerate eigenfunctions at the critical point. This indicates the appearance of a pair of soft modes, resulting in a transverse instability of the optimal escape trajectory as it approaches the saddle.

Computation of the determinant quotient in Eq. (13) is less straightforward when the transition state is nonconstant. Moreover, the rotational degeneracy of the nonconstant state (corresponding to the fact that the domain wall pair can be localized with uniform probability at any $\theta_0$) implies a soft collective mode in the linearized dynamical operator $\hat{\mathbf{A}}_u$ of Eq. (13). The result is a zero eigenvalue for all $h < h_c(\ell)$. This has a different physical origin from the vanishing exactly at $h_c(\ell)$ of the lowest stable eigenvalue of the saddle(s), and leads to a temperature dependence of the prefactor — and hence to non-Arrhenius behavior of the overall transition rate — in the entire $(\ell, h)$ region above the phase boundary.

This behavior emerges in the following way. Computation of the fluctuation determinant requires removal of the zero eigenvalue arising from the soft collective mode, and can be accomplished using the McKane-Tarlie regularization procedure. Its removal leads to an additional factor of $2\ell \sqrt{|y_1|y_1}/\pi k_B T$, which is the mathematical origin of the temperature dependence of the prefactor.

Removal of the zero eigenvalue has another effect as well, namely a cancellation of the divergence of the prefactor as the phase boundary is approached from the ‘instanton side’. The term $|y_1|y_1$ in the above factor is the square of the norm of the zero mode, which vanishes as $m \to 0$, in such a way as to exactly remove the divergence caused by the vanishing of the lowest stable eigenvalue of the instanton state.

We still need to show that the instanton state is the transition state above the phase boundary in the $(\ell, h)$ plane. The eigenfunction $y_1 \propto \text{sn}(\lambda(m)) \text{cn}(\lambda(m)) \text{cn}^2(\ell/m) + \text{sn}^2(\ell/m) \text{dn}^2(\lambda(m))$ has a single pair of nodes. Because nodes arise in pairs (due to the periodic boundary conditions), there is consequently only a single (nodeless) solution of lower eigenvalue than $y_1$. But because $y_1$ is the eigenfunction with zero eigenvalue, we have shown that the solution (8) has a single unstable eigenmode, and is therefore a proper saddle.

The formula for the prefactor per unit length in the region in which the instanton solution is the saddle is then given by

$$\tau_0 \Gamma_0^\dagger / \ell = C m(k_B T)^{-1/2} \sinh \left[ \delta \sqrt{1 - h K(m)} \right]$$

$$ \times \left[ 2mK(m) \frac{d \log \delta}{dm} + \frac{1}{1 - m} \frac{E(m) - (1 - m)K(m)}{K(m)} \right]^{-1/2},$$

(15)

where $E(m)$ is the complete elliptic function of the second kind and $C$ is a weakly $h$- and $\ell$-dependent ‘constant’ (i.e., independent of $T$) that is $O(1)$ everywhere.

Summarizing, there are two important physical consequences of the the zero eigenvalue arising from the uniform translation mode. The first is the presence in the switching rate of non-Arrhenius behavior — i.e., a $T$-dependent prefactor — everywhere on the low-field side of the transition, as can be seen in Fig. 4.

The second consequence (which would be harder to see experimentally) is the removal of the divergence of the prefactor on the instanton side of the critical boundary. The factor of $m$ in the numerator arises from the norm of the zero mode, as explained above. As $m \to 0$, this term exactly cancels the divergence of the determinant ratio as the critical point is approached: $\lim_{m \to 0} \tau_0 \Gamma_0^\dagger \sqrt{k_B T} / \ell \to \sqrt{2}(1 - h^2)^{9/4} \sinh[\pi/\sqrt{1 + h}]/(\pi^2 \sqrt{h^2 + 1/2})$. 


Figure 4. Total switching rate (in units of $s^{-1}$) vs. $\beta = 1/k_B T$ (in units of $^\circ{K}^{-1}$), at fields of (a) 52.5 mT (instanton saddle) and (b) 72.5 mT (constant saddle). Parameters used are $k = .01$, $l = .05$, $R = 200$ nm, $R_1 = 180$ nm, $R_2 = 220$ nm, $M_0 = 8 \times 10^5$ A/m (permalloy), $\alpha = .01$, and $\gamma = 1.7 \times 10^{11} T^{-1} s^{-1}$. Deviation of low-field switching rate in (a) from dashed line signals non-Arrhenius behavior. (From Martens et al.\textsuperscript{33})

A detailed discussion of the meaning and interpretation of the prefactor divergence is presented in Stein.\textsuperscript{19} Near (but not at) the critical point the prefactor formulae hold, but in a vanishing range of $T$ as $\ell_c$ is approached. Exactly at $\ell_c$ the prefactor is finite but non-Arrhenius (with a different exponent than that in Eq. (15)). If one instead holds $T$ fixed (but still small compared to $\Delta W$) and crosses the critical boundary (say, by varying $h$ at fixed $\ell$), then the transition is rounded.

6. SUMMARY

A theory of magnetization reversal in thin mesoscopic rings has been presented. Such systems are especially stable against magnetic reversal, due to their lack of edges or corners where nucleation is easily initiated. The two-dimensional annulus geometry also affords a separation of long-range magnetostatic forces into local and nonlocal terms.\textsuperscript{36} Because the nonlocal terms in this system are smaller than the local ones, an analytic treatment is possible and a variety of new and interesting phenomena can be predicted and understood.

Nevertheless, the nonlocal terms are not altogether ignorable. According to the KS analysis, the nonlocal (bulk magnetostatic) contribution to the total magnetic energy is separated from the local (boundary magnetostatic) term only by a logarithm. Therefore, even for many presently attainable ring geometries, these terms will generally make some nonnegligible contribution to the total energy. On the other hand, ring thicknesses and radii are now becoming available in which the bulk term is roughly an order of magnitude (or even more) smaller than the other terms. In these systems the bulk magnetostatic contribution can then be viewed as a minor (if nonlocal) adjustment of the exchange term, and the analysis presented here should capture the essential physics of magnetic reversal.

Among the most interesting of the new phenomena that emerge from this analysis is an unusual transition in activation behavior.\textsuperscript{18, 19} Our results suggest that such a transition should now be observable experimentally, by varying the externally applied magnetic field for rings of fixed size. That the transition not only exists, but also is a function of field strength $h$ for fixed ring radius $R$, is a major qualitative result of this paper. It affords the first accessible experimental test of the transition in classical activation behavior uncovered by MS.\textsuperscript{18} A clear signature of this transition would be observation of Arrhenius behavior of the magnetic reversal at high fields and non-Arrhenius behavior at low fields, as in Fig. 4. This effect is small but should be observable in the statistics of numerous runs where reversal occurs.
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38. The magnetostatic energy also contains a mixing term between bulk and boundary contributions, which scales the same as the bulk term; it is therefore omitted here. Its contribution is the same order as that of the bulk term, which is shown in the text to be at least an order of magnitude smaller than all of the other terms.